Algebra-I
B. Math - First year
Backpaper Exam 2012-2013

Time: 3hrs
Max score: 100

All questions carry equal marks. Answer all.

(1) Let $\phi : G \longrightarrow G$ be an automorphism of a finite group G satisfying $\phi(g) = g$ if and only if $g = e$.
(a) Prove that every element of G must be of the form $x^{-1}\phi(x)$ for some $x \in G$.
(b) If in addition, $\phi^2 = \mathrm{id}$ then show that $\phi(x) = x^{-1}$ for all $x \in G$. Hence show that G is abelian. 8+12

(2) (a) Let H be a normal subgroup of a group G. Show that

$$\begin{aligned} \phi : \quad & G \times H \longrightarrow H \\ & (g, h) \longmapsto ghg^{-1} \end{aligned}$$

defines an action of G on H.
(b) Using the permutation representation of the above action show that $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.
(c) Let $o(G) = 3825$. Prove that if H is a normal subgroup of order 17 in G then $H \subseteq Z(G)$. 2+8+10

(3) (a) Show that two elements of S_n are conjugate in S_n if and only if they have the same cycle type.
(b) Compute the order of the centralizer $C_{S_n}(\sigma)$, where $\sigma = (12)(34)$ in S_n. 10+10

(4) (a) State the three Sylow's theorems.
(b) Let $o(G) = 105$. Prove that if a Sylow 3-subgroup is normal then G is abelian. 8+12

(5) (a) Show that an abelian group of order pq, where p, q are distinct primes, is cyclic.
(b) Show that a group of order p^2q, p and q distinct primes is not simple. 10+10